

# Unions of regular polygons with large perimeter-to-area ratio

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## Abstract

T. Keleti [1] asked, whether the ratio of the perimeter and the area of a finite union of unit squares is always at most 4. In this paper we present an example where the ratio is greater than 4.

## 1 Introduction

Tamás Keleti [1] proved that if we take a finite union of unit squares, then the perimeter-to-area ratio of the union cannot be arbitrarily large. In fact, he proved a general result concerning so called  $r$ -star-shaped sets in  $\mathbb{R}^n$  (in particular every compact convex set): the perimeter to the area ratio of a finite union of congruent  $r$ -star-shaped sets is bounded.

One would be tempted to think that the upper bound is realised by a single set, however this is not the case, even if we consider solely compact convex polygons. Gyenes [3] gave an example, where the perimeter-to-area ratio of the union of two polygons exceeds the perimeter-to-area ratio of a single one. On the other hand for circles this statement holds true.

So it is very natural to ask the following:

**Question 1.1.** (*Keleti*) *Is it true that the perimeter-to-area ratio of a single regular  $n$ -gon with side length 1 maximises the perimeter-to-area ratio of the union of regular  $n$ -gons with side length 1?*

Gyenes gave a new proof for the boundedness of the ratio, improving the upper bound to 5.6 in the case of squares. Also, he proved that the upper bound is 4, if we consider squares with common centre or with sides parallel to the axis, or if we consider the union of two squares (see [2],[3]).

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P. Humke, C. Marcott, B. Mellem and C. Stiegler investigated the differentiation properties of the perimeter and area functions related to Keleti's question [4].

In this paper we give a negative answer to this question for  $n = 3$  and  $n = 4$ , give some examples with large perimeter-to-area ratio for  $n = 4$  and finally we list a number of open problems. The idea of the counterexamples is motivated by the results of a probabilistic computer algorithm. In our experiments, we used the open source JTS Topology Suite library.

## 2 Results

First we need some technical definitions.

**Definition 2.1.** Let  $n$  be a natural number. We call a *basic  $(k, n)$ -setup*  $k$  many regular  $n$ -gons with origin centre and side length 1 with vertices rotated to form a regular  $kn$ -gon.

**Definition 2.2.** Suppose that we have a finite union of regular polygons and  $v \in \mathbb{R}^2$ . We call the translation by  $v$  of one of the polygons  $P$  *small enough*, if for every  $t \in [0, 1]$  the translation of  $P$  by  $tv$  does not change the pattern of the intersections on the boundary. The translation is called *regular*, if  $v$  is parallel to the vector from the centre of  $P$  to one of the vertices of  $P$ .

We will denote the area of a polygon  $P$  by  $a(P)$ , and the perimeter by  $p(P)$ .

### 2.1 A counterexample of 5 squares

**Theorem 2.3.** *There exists a polygon that is the union of 5 squares with side length 1 for which the perimeter-to-area ratio is greater than 4.*

First we reiterate the proof of Gyenes for a basic  $(k, n)$ -setup.

**Lemma 2.4.** *The perimeter-to-area ratio of the union of the  $n$ -gons in a basic  $(k, n)$ -setup is equal to the perimeter-to-area ratio of a single  $n$ -gon.*

*Proof.* The union is a polygon, let us denote the vertices by  $A_0 A_1 \dots A_l$  and the distance of the line segment  $AB$  from the origin by  $d_{AB}$ . Now clearly  $a(A_0 A_1 \dots A_l) = \frac{1}{2}(|A_0 A_1| d_{A_0 A_1} + |A_1 A_2| d_{A_1 A_2} + \dots + |A_l A_0| d_{A_l A_0})$ . But since  $A_i A_{i+1}$  are segments which are subsets of the boundaries of congruent regular  $n$ -gons with origin centre, we have that  $d_{A_0 A_1} = d_{A_1 A_2} = \dots = d_{A_l A_0} = d$ . So

$$\frac{p(A_0 A_1 \dots A_l)}{a(A_0 A_1 \dots A_l)} = \frac{2}{d}$$

which is the same as in the case of a single  $n$ -gon. □

Now we begin with a basic  $(5, 4)$ -setup and start to move the polygons away from the origin. Each polygon will be moved by a regular small enough translation in such a way that the centres will form a regular pentagon around the origin, as shown in the figure. We will prove that the perimeter will remain the same, while the area will be less then the original.

**Lemma 2.5.** *The perimeter will remain the same after shifting the squares.*

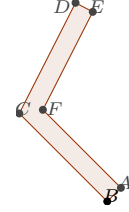
*Proof.* First, shift one square only. The square has eight segments which are parts of the boundary. The length of the two segments at the vertex the square is shifted to, increase with the same amount as the length of the segments at the opposite vertex decrease with.

At the other two vertex, the sum of the length of the segments will be constant, since one will increase with the same amount as the other decreases.

We can shift the squares one by one, and the same can be said about every square, so the perimeter remains the same.  $\square$

**Lemma 2.6.** *The area of the union decreases as we shift the squares.*

*Proof.* Let  $\varepsilon$  be the size of the shift and let  $x$  be the common length of the segments on the boundary of the original construction, with centres at the origin. Let  $T$  be area of the hexagon  $ABCDEF$ , where  $EF = FA = x$ ,  $AB = DE = \varepsilon/\sqrt{2}$ , the angles at  $A$ ,  $B$ ,  $D$  and  $E$  are right angles, and the angle at  $C$  is  $108^\circ$ . Let  $t$  be the area of the same hexagon, only this time the length of the sides  $BC$  and  $CD$  will equal  $x$ . It is easy to see, that  $t < T$ .



Because of the symmetry of the construction, the area added to the union is exactly ten times the area of the polygon  $A_4A'_4PE'_1QE$  which is  $t - \frac{\varepsilon^2}{4}$  since  $A_4R = RE_1 = x$ . The area subtracted from the union is ten times the area of the hexagon  $C_2C'_2VUB_3T$  which is  $T - \frac{\varepsilon^2}{4}$ , because  $C_2T = TB_3 = x$ . So the area subtracted is bigger than the area added, which finishes the proof.  $\square$

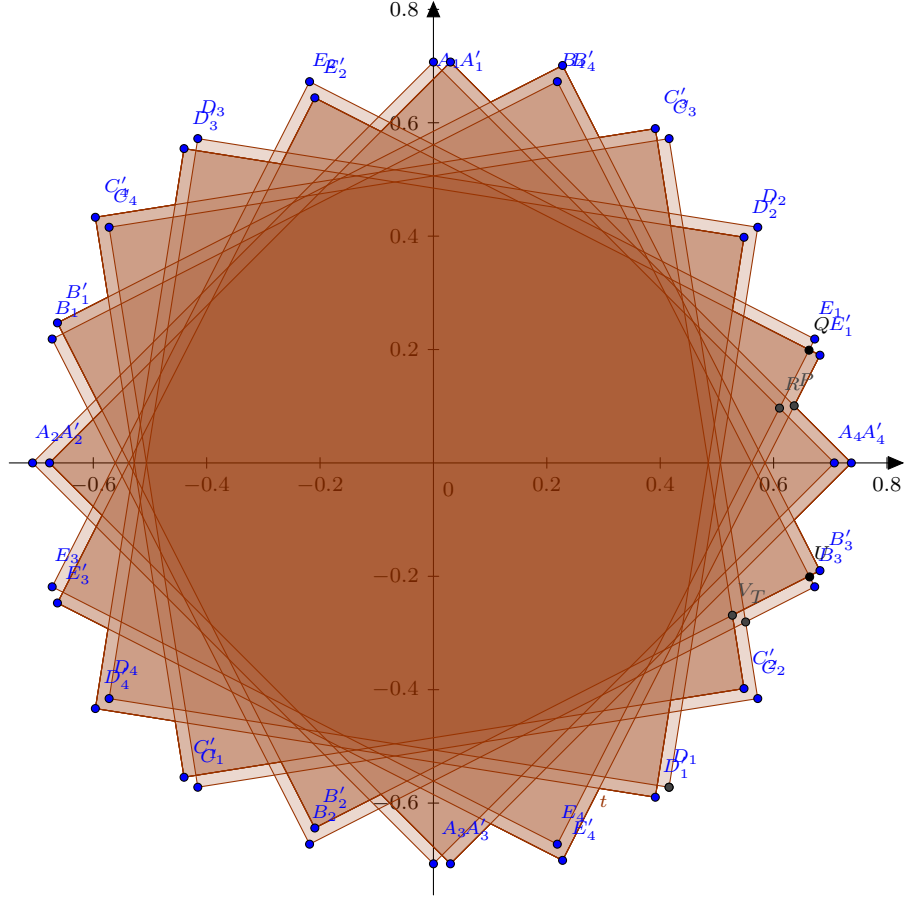


Figure 1: The squares shifted from the basic (5,4) setup

## 2.2 A counterexample of 4 regular triangles

**Theorem 2.7.** *There exists a polygon that is the union of 4 regular triangles with side length 1 for which the perimeter-to-area ratio is greater then  $4\sqrt{3}$  (which is the perimeter-to-area ratio of a single triangle).*

*Proof.* The idea is the similar as before, we start with a basic (4,3)-setup. In this case the calculation is particularly convenient if we translate solely two of the triangles, since the translation will not change the perimeter of the union, however the area will decrease.

For the sake of exactitude we prove the theorem through 3 easy lemmas.

**Lemma 2.8.** *Suppose that we have 4 triangles obtained by small enough translations of the basic setup. Then if we apply a small enough regular translation to one of the triangles then it does not change the perimeter of the union.*

*Proof.* W.l.o.g. we can assume that we translate a triangle to the positive direction of the  $y$  axis, as in Figure 2.

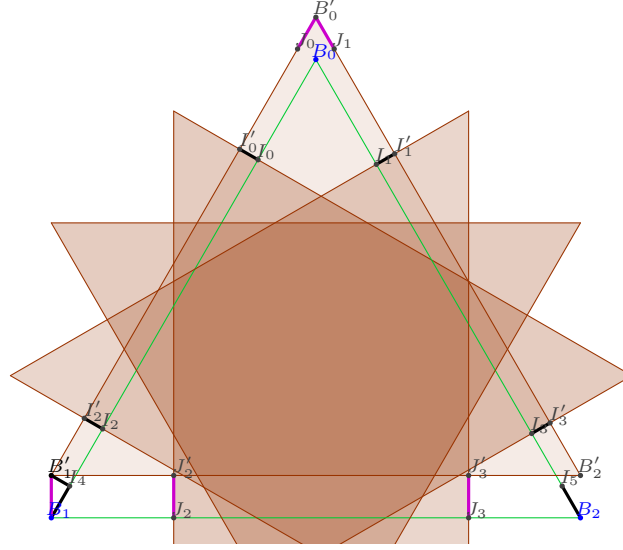


Figure 2: Calculation of the perimeter

Clearly, the perimeter of the union decreases with the length of  $I_0I'_0, \dots, I_3I'_3$  and  $I_4B_1$  and  $I_5B_2$ , where  $I_4$  and  $I_5$  are assigned so that the triangles  $B'_1I_4B_1$  and  $B'_2I_5B_2$  are right-angled. The increase of the perimeter is equal to the sum of the lengths of  $J_0B'_0$ ,  $J_1B'_0$  and  $J_2J'_2$ ,  $J_3J'_3$ .

Now notice that the triangle  $B'_1I_4B_1$  has angles  $60, 90, 30$  respectively and  $|B'_1I_4| = |I_0I'_0| = |I_2I'_2|$ , and  $|B'_1B_1| = |J_2J'_2|$ . Therefore  $|J_2J'_2| = |I_0I'_0| + |I_2I'_2|$ . The triangles  $B'_1I_4B_1$  and  $B_0J_0B'_0$  are congruent, so  $|I_4B_1| = |J_0B'_0|$ . Similarly  $|J_3J'_3| = |I_1I'_1| + |I_3I'_3|$  and  $|I_5B_2| = |J_1B'_0|$ .

Thus, the increase equals to the decrease so the translation does not change the perimeter.  $\square$

**Lemma 2.9.** *Suppose that we have 4 triangles in the basic (4,3)-setup. If we apply a small enough regular translation to one of the triangles then it does not change the area of the union.*

*Proof.* Again, w.l.o.g. we can assume that we have translated the triangle  $B_0B_1B_2$  with the vector  $(0, \varepsilon)$  and we obtain the triangle  $B'_0B'_1B'_2$  as shown in Figure 3. According to the notation of the figure, it is clear that the change in the area is equal to  $2(a(B_0B'_0I'_0I_0) + a(I'_2I_2L_1B'_1) - a(L_1J'_2J_2B_1))$  (i.e. the area of the red and pink figures is added and the area of the green figures is subtracted). Let us denote  $|B_1J_2|$  by  $d$ . Now we calculate this value.

Clearly,

$$a(L_1J'_2J_2B_1) = \frac{d + d - \varepsilon/\sqrt{3}}{2} \cdot \varepsilon$$

and by symmetry  $|B_1I_2| = |B_0I_0| = d$  and we have  $|I'_2I_2| = |I'_0I_0| = \varepsilon/2$ .

$$a(B_0B'_0I'_0I_0) = \frac{d + d + (\sqrt{3}/2)\varepsilon}{2} \cdot \frac{\varepsilon}{2}$$

and

$$a(I'_2I_2L_1B'_1) = \frac{d - (2/\sqrt{3})\varepsilon + d - \varepsilon/(2\sqrt{3})}{2} \cdot \frac{\varepsilon}{2}.$$

Adding up this equalities gives the lemma. □

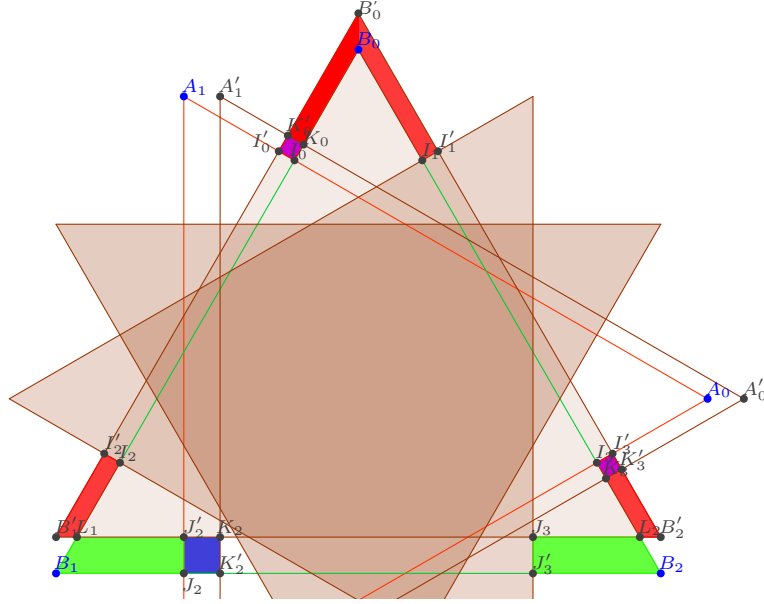


Figure 3: Calculation of the area

**Lemma 2.10.** *Suppose that we translate away two different neighbouring triangles from the basic setup by small enough regular non-identical translations. Then the area of the union of the triangles decreases.*

*Proof.* By the previous lemma we have that the translation of one triangle does not change the area, so to obtain the difference of the area, it is enough to consider the modifications what the translation of the first triangle gives (the blue and pink rectangles on the figure). In our case it will decrease the area. To be precise, let the translated triangles be  $A_0A_1A_2$  and  $B_0B_1B_2$  and the translations are  $(\delta, 0)$  and  $(\varepsilon, 0)$  (Figure 3). Then suppose that we have translated  $A_0A_1A_2$  to  $A'_0A'_1A'_2$  'first'. Now the translation of  $B_0B_1B_2$  to  $B'_0B'_1B'_2$  increases the area of the union by

$$a(B'_0B_0K_0K'_0) + a(I'_2I_2L_1B'_1) + a(B'_2L_2K_3K'_3) + a(B'_0B_0I_1I'_1)$$

and decreases it by  $a(B_1L_1K_2K'_2) + a(J_3L_2B_2J'_3)$  (the quadrilaterals added coloured by red, the subtracted ones coloured by green and blue on Figure 3).

Thus we have

$$\begin{aligned}
& a(B'_0 B_0 K_0 K'_0) + a(I'_2 I_2 L_1 B'_1) + a(B'_2 L_2 K_3 K'_3) + a(B'_0 B_0 I_1 I'_1) \\
& \quad - a(B_1 L_1 K_2 K'_2) - a(J_3 L_2 B_2 J'_3) \\
& < a(B'_0 B_0 I_0 I'_0) + a(I'_2 I_2 L_1 B'_1) + a(B'_2 L_2 I_3 I'_3) + A(B'_0 B_0 I_1 I'_1) \\
& \quad - a(B_1 L_1 J_2 J'_2) - a(J_3 L_2 B_2 J'_3) = 0.
\end{aligned}$$

□

Putting together the 3 lemmas we have that after translating two different neighbouring triangles by regular small enough translations, the area decreases, but the perimeter does not change. Since by Lemme 2.4 the perimeter-to-area ratio of the basic (4, 3)-setup equals to  $4\sqrt{3}$ , we are done. □

### 3 Other counterexamples

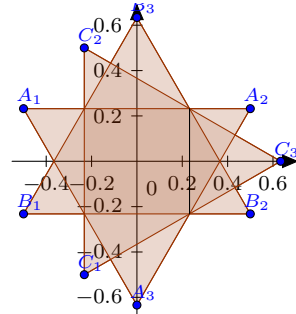
Let us mention that taking squares with common centre and translate away and rotate them slightly works for 4 squares as well. Without proof we present a construction of four unit squares. The four squares will be the following:

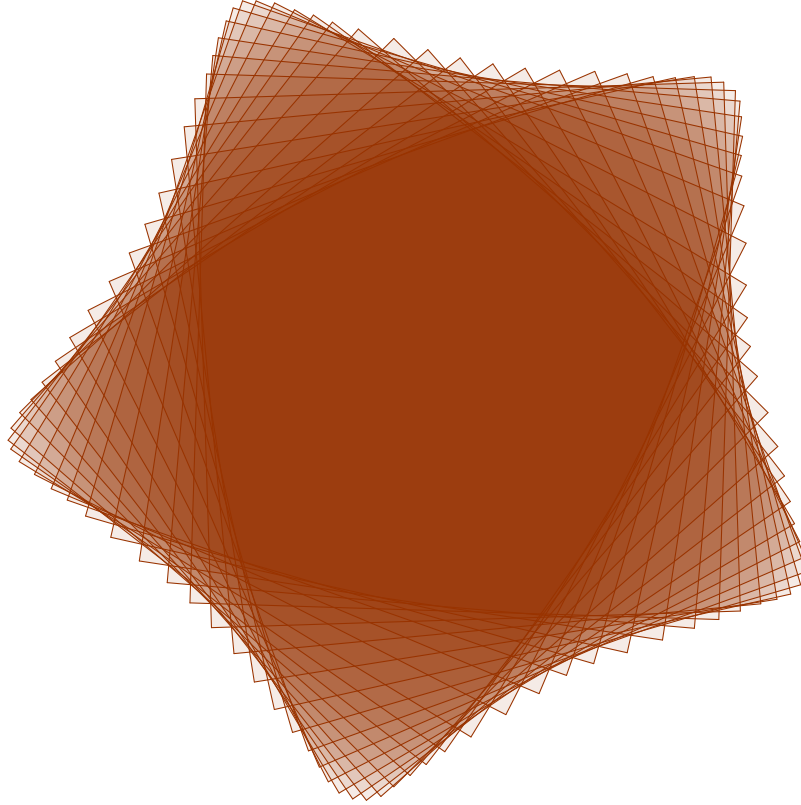
$$\begin{aligned}
S_1 &= \text{conv} \left\{ \left( \frac{1}{2}, \frac{1}{2} \right), \left( -\frac{1}{2}, \frac{1}{2} \right), \left( -\frac{1}{2}, -\frac{1}{2} \right), \left( \frac{1}{2}, -\frac{1}{2} \right) \right\}, \\
S_2 &= \text{conv} \left\{ \left( \frac{149}{650}, \frac{399}{650} \right), \left( -\frac{451}{650}, \frac{149}{650} \right), \left( -\frac{201}{650}, -\frac{451}{650} \right), \left( \frac{399}{650}, -\frac{201}{650} \right) \right\}, \\
S_3 &= \text{conv} \left\{ \left( \frac{399}{650}, \frac{201}{650} \right), \left( -\frac{201}{650}, \frac{451}{650} \right), \left( -\frac{451}{650}, -\frac{149}{650} \right), \left( \frac{149}{650}, -\frac{399}{650} \right) \right\}, \\
S_4 &= \text{conv} \left\{ \left( -\frac{91}{1450}, \frac{41}{58} \right), \left( -\frac{1141}{1450}, \frac{1}{58} \right), \left( -\frac{141}{1450}, -\frac{41}{58} \right), \left( \frac{909}{1450}, -\frac{1}{58} \right) \right\}.
\end{aligned}$$

In the case of triangles we do not need 4 to a form a counterexample. We sketch the construction for 3 triangles.

Let  $S$  be the biggest square that can be written into a unit equilateral triangle, with one line segment in common with the triangle. Now draw three equilateral triangle around  $S$ , with three sides of the square lying on one-one side of the triangles. This construction works. Actually, it works even if we add the forth triangle to the figure.

Using a standard optimisation algorithm, we found the following figure of a construction containing 25 squares with ratio of about 4.28.





## 4 Open problems

We believe that the constructions containing five squares and four equilateral triangles can be generalised to give a counterexample of  $n + 1$  regular polygons with  $n$  vertices and unit side length.

Let us call a  $k$  many regular  $n$ -gon with unit side length be given with centres at the vertices  $C_1, C_2, \dots, C_k$  of a small regular  $k$ -gon around the origin. The vertex of the  $i$ -th polygon farthest from the origin should be on the line given by the origin and  $C_i$ . We call this a *shifted  $(k, n)$ -setup*.

**Conjecture 4.1.** *This construction works, i.e. the perimeter-to-area ratio of a shifted  $(n + 1, n)$ -setup is greater than the ratio in the case of a single regular  $n$ -gon with unit side length.*

A general question consistent with the results obtained by computer is the following:

**Question 4.2.** *Is it true that a shifted  $(k, n)$ -setup yields a counterexample iff  $k > 1$  and  $k \equiv 1 \pmod{n}$ ?*



Since Keleti's boundedness result works in  $\mathbb{R}^l$  as well it is also natural to ask the following:

**Question 4.3.** *Do the analogous constructions work in higher dimensions for regular polyhedrons?*

We have found a counterexample using four squares, but could not find any using only three.

**Question 4.4.** *What is the minimum number of squares that form a counterexample? Or in general, what is the minimum number of regular  $n$ -gons to form a counterexample? Is it equal to  $n$ ?*

Since we have found the first counterexamples with a probabilistic computer algorithm, it would be interesting to know that whether the 'majority' of the setups close to the basic setup is a counterexample in some sense. For  $k$ -many regular  $n$ -gons let us denote the centres by  $C_1, C_2, \dots, C_k$  and the rotations of the polygons by  $r_1, \dots, r_k$  respectively. Let  $f_{k,n}(C_1, \dots, C_k, r_1, \dots, r_k)$  be the perimeter-to-area ratio of such a setup and  $p_0 = (C_1^0, \dots, C_k^0, r_1^0, \dots, r_k^0)$  the centres and rotations of the basic  $(k, n)$ -setup (of course,  $C_i^0 = (0, 0)$ ).

**Question 4.5.** *What are the derivatives of the function  $f_{k,n}$  at the point  $p_0$ ? Is this point a local minimum of  $f_{k,n}$ ? Or does there exist a neighbourhood of  $p_0$  where almost all points are counterexamples?*

We also could not go close to the current best upper bound of about 5.6 proved for the ratio by Gyenes. The best ratio we could find with the help of a computer has ratio about 4.34 and contains 100 squares.

**Question 4.6.** *What is the optimal upper bound for the ratio?*

## References

- [1] T. Keleti, A covering property of some classes of sets in  $\mathbb{R}^n$ , *Acta Univ. Carolin. Math. Phys.* **39** (1998), no. 1-2, 111–118.
- [2] Z. Gyenes, The ratio of the perimeter and the area of unions of copies of a fixed set, *Discrete Comput. Geom.* **45** (2011), no. 3, 400–409.
- [3] Z. Gyenes, The ratio of the surface-area and volume of finite union of copies of a fixed set in  $\mathbb{R}^n$ , *MSc thesis*, 2005
- [4] P. Humke, C. Marcott, B. Mellem, C. Stiegler, Differentiation Properties Related to the Keleti Perimeter to Area Conjecture, *Erdős Centennial, conference presentation*, 2013